

# Aggregation–Fragmentation Processes and Wave Kinetics

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There is a formal correspondence between the isotropic 3-wave kinetic equation and the rate equations for a non-linear fragmentation–aggregation process. We exploit this correspondence to study analytically the time evolution of the wave frequency power spectrum. Specifically, we analyzed a 3-wave turbulence in which the wave interaction kernel is a constant. We consider both forced and decaying turbulence. In the forced case, the scaling function diverges as  $x^{-3/2}$  as expected from Kolmogorov–Zakharov theory. In the decaying case, the scaling function exhibits non-trivial, and hitherto unexpected, divergence with both algebraic and logarithmic spectral exponents which we calculate. This divergence leads to non-trivial decay laws for the total wave action and the number of primary waves. All theoretical predictions are verified with high quality numerical simulations of the 3-wave kinetic equation.

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Wave turbulence is a theory of the statistical evolution of ensembles of weakly nonlinear dispersive waves. It has been applied to capillary waves on fluid interfaces, gravity waves on the ocean, acoustic turbulence and various special limits of plasma and geophysical turbulence. (For a review of the theory see [1]; for a summary of applications see [2].) The key feature is the fact that weak nonlinearity permits the consistent derivation [3] of a wave kinetic equation describing the time evolution of the frequency power spectrum,  $N_\omega(t)$ . When sources and sinks of energy, widely separated in frequency, are added to the wave kinetic equation, it can be shown to have exact stationary solutions corresponding to a cascade of energy through frequency space from the source to the sink. The cascade solution is known as the Kolmogorov–Zakharov (K-Z) spectrum; it describes an intrinsically non-equilibrium state of the wave field. Everything is known about the stationary K-Z spectra, their scaling exponents, locality and stability. By contrast, very little is known about the time-dependent solutions of the wave kinetic equation. A basic scaling theory of the development of the stationary state in the case of forced wave turbulence was provided in [4] although numerical investigations [5, 6] have suggested that there are unexplained dynamical scaling anomalies in many cases. Almost nothing is known about time-dependent solutions in the case of decaying turbulence where an initial spectrum is allowed to decay in the absence of external forcing. In this Letter we take the first steps to remedy this.

The subject of aggregation–fragmentation kinetics, having its origins in theoretical chemistry has, at first sight, rather little to do with waves or turbulence. This field concerns itself with the statistical mechanics of ensembles of particles which aggregate or fragment upon contact. The principal quantity of interest is the particle

size distribution,  $n_i(t)$ , denoting the density of clusters of mass  $i$  at time  $t$ . It satisfies a kinetic equation, which, in the case of pure aggregation, is the well-known Smoluchowski coagulation equation [7]. For a review of pure aggregation processes see [8]. If clusters also break up, additional terms may be added to the Smoluchowski equation to take this into account. See [9] for a review of fragmentation. In aggregation–fragmentation kinetics, in strong contrast with wave kinetics, almost all theoretical effort has historically been focused on determining the time evolution of  $n_i(t)$  from the underlying kinetic equation. As a result, a comprehensive scaling theory of the solutions of the Smoluchowski equation has been constructed (see [10] for a review). Although there is a conceptual analogy [11] between energy transfer between scales in turbulence and mass transfer between clusters in aggregation, it is only recently that this analogy has been made quantitatively useful. Concepts and techniques from turbulence have proven useful in analysing certain aspects of aggregation problems [12, 13, 14]. Furthermore, it has been shown [15] that, in the case of isotropic wave turbulence with quadratic nonlinearity, the wave kinetic equation can be rewritten as a set of rate equations for an aggregation-fragmentation process with an unusual non-linear fragmentation mechanism. This correspondence opens the door for the transfer of ideas and techniques from aggregation–fragmentation kinetics to the context of wave turbulence which will hopefully start to fill in the gap in knowledge of time-dependent solutions of the wave kinetic equations alluded to already. Furthermore, this correspondence opens up a new set of problems within aggregation–fragmentation kinetics. This Letter contains some opening explorations in this direction.

It was shown in [15] that resonant interactions between waves lead to forward transfer of energy between frequen-

cies which looks like an aggregation process:  $(i) \oplus (j) \rightarrow (i + j)$ . Back-scatter of energy leads to a fragmentation process  $(i) \oplus (i + j) \rightarrow (i) \oplus (i) \oplus (j)$ . This fragmentation mechanism is unusual. It is non-linear while typically [9] the fragmentation mechanism is linear:  $(i + j) \rightarrow (i) \oplus (j)$ . Nonlinear collision-controlled fragmentation processes have been studied in the past (see [16] and references therein). While they are somewhat similar to the above rule, this model asserts that only the larger particle breaks and it happens according to a rather special rule. Our goal here is to apply ideas and techniques developed in studies of aggregation and fragmentation to wave turbulence. We want to examine fundamental aspects and have, therefore, limited ourselves to the simplest possible situation where the wave interaction kernels are constant. The dynamical problem is already non-trivial at this level. The resulting kinetic equation can be reduced (see [15] for details), in the discrete case [18], to:

$$\begin{aligned} \frac{dN_\omega}{dt} = & J \delta_{\omega 1} + \frac{1}{2} \sum_{\omega_1 + \omega_2 = \omega} N_{\omega_1} N_{\omega_2} - N_\omega \sum_{\omega_1 \geq 1} N_{\omega_1} \\ & - N_\omega \sum_{\omega_1 < \omega} N_{\omega_1} + N_\omega \sum_{\omega_1 > \omega} N_{\omega_1} + \sum_{\omega_1 \geq 1} N_{\omega_1} N_{\omega + \omega_1} \end{aligned} \quad (1)$$

where  $J$  is the energy injection rate and  $N_\omega$  is the frequency space wave action. The total wave action is  $N(t) = \sum_{\omega \geq 1} N_\omega(t)$ . In the decay case ( $J = 0$ ), it satisfies the equation (found by summing Eqs. (1))

$$\frac{dN}{dt} = -\frac{1}{2} \sum_{\omega \geq 1} N_\omega^2. \quad (2)$$

The primary waves (monomers) evolve according to

$$\frac{dN_1}{dt} = -N_1^2 + \sum_{\omega \geq 1} N_\omega N_{\omega+1}. \quad (3)$$

We assume the *scaling hypothesis*: there exists a typical scale,  $s(t)$ , such that  $N_\omega(t)$  is asymptotically of the form

$$N_\omega(t) = s^a F(\omega/s). \quad (4)$$

Given this hypothesis, it follows from Eq. (1) that

$$\frac{ds}{dt} = s^{a+2} \quad (5)$$

while  $F(x)$  must satisfy a complicated integro-differential equation. The scale  $s(t)$  is defined as a ratio of moments:

$$s(t) = \frac{M_2(t)}{M_1(t)} \quad M_n(t) = \int_0^\infty \omega^n N_\omega(t) d\omega \quad (6)$$

Often the scaling function,  $F(x)$ , diverges at small  $x$ :

$$F(x) \sim A x^{-\tau} \text{ as } x \rightarrow 0. \quad (7)$$

The exponent  $\tau$  is the wave spectrum exponent or polydispersity exponent. The shape of the frequency power spectrum for large time is determined by the small  $x$  behaviour of the scaling function,  $F(x)$ . In aggregation problems this divergence has been often encountered and the  $\tau$  has proven to be difficult to determine [10, 17]; in some seemingly simple models the exponent  $\tau$  remains unknown. An important lesson from this work is that one should be particularly careful when  $\tau \geq 1$ .

This is a finite capacity system so there is no dissipative anomaly / gelation transition. Energy is therefore conserved for all time by the wave interactions. For the forced case, the total energy then grows linearly in time since we are injecting energy at a constant rate. Thus  $M_1 \sim t$  (we take  $J = 1$ ). The scaling hypothesis, Eq. (4), then implies that  $a = -\frac{3}{2}$  and subsequently solving Eq. (5) suggests that  $s(t) \sim r_0 t^2$  for some constant,  $r_0$ . We then expect the scaling

$$N_\omega(t) \sim s^{-3/2} F(\omega/s) \text{ with } s \sim r_0 t^2. \quad (8)$$

On the other hand, for the decaying turbulence energy is conserved,  $M_1(t) = 1$ . The scaling hypothesis then implies that  $a = -2$  and solving Eq. (5) gives  $s(t) \sim s_0 t$  for some constant,  $s_0$ . We then expect the scaling

$$N_\omega(t) \sim s^{-2} F(\omega/s) \text{ with } s \sim s_0 t. \quad (9)$$

These predictions for the growth rate of  $s(t)$ , which are based solely on the assumption of scaling and the absence of a dissipative anomaly, are verified numerically in Fig. 1. All numerics have been done using the algorithm described in [15].

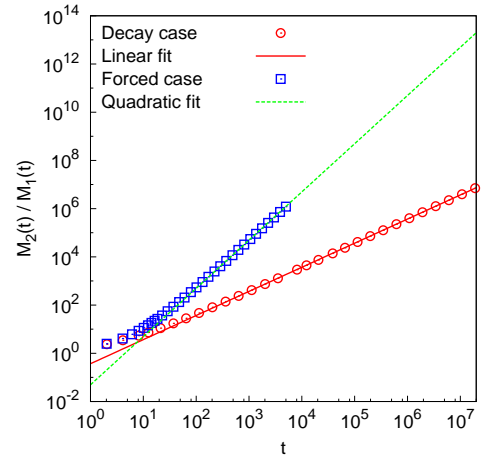


FIG. 1: Time evolution of typical frequency, Eq. (6), for forced and decaying turbulence. The dashed lines correspond to the theoretical predictions provided by Eq. (8) and Eq. (9) respectively

For forced turbulence, we expect that the frequency spectrum should become stationary for large times. This

is an additional piece of information which allows us to fix the spectral exponent. Requiring that Eq. (8) is independent of  $t$  for small  $\omega$  selects  $\tau = 3/2$ . This corresponds to the K-Z exponent for this model [15]. Furthermore, the corresponding K-Z constant can be computed exactly for this model [15] so that we obtain asymptotic behaviour of the scaling function

$$F(x) \sim \frac{x^{-\frac{3}{2}}}{2\sqrt{\pi - 4\ln 2}} \text{ as } x \rightarrow 0. \quad (10)$$

This prediction, and the scaling behaviour, Eq. (8) are verified explicitly from the numerical data in Fig. 2.

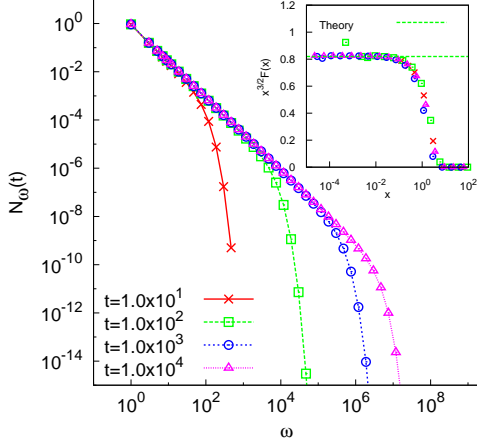


FIG. 2: Time evolution of wave spectrum in the forced case. The main panel shows snapshots of  $N_\omega(t)$  at a succession of times. The inset shows the same data collapsed according to the scaling in Eq. (8). The collapsed data has been compensated by  $x^{3/2}$  in accordance with Eq. (10) and shows a plateau with the theoretically predicted amplitude.

Let us now turn to the decay case. Figure 3 presents numerical simulations of the decay of a monochromatic initial spectrum  $N_\omega(0) = \delta_{\omega_1}$  and verifies the scaling behaviour expected from Eq. (9). The essential difference from the forced case is that we no longer have the additional constraint provided by stationarity which allowed us to easily determine the spectral exponent  $\tau$ . We must return to the original kinetic equation.

Guided by our result for the forced case, let us *presume* that the wave spectrum diverges as  $x \rightarrow 0$  in accordance with Eq. (7). Substitution of Eq. (4) into Eq. (3) and comparing requires us to choose:  $\tau - 3 = 2\tau - 4$  and  $A = \frac{2-\tau}{1-\sum_{\omega \geq 1} \frac{1}{[\omega(\omega+1)]^\tau}}$ . This seems to straightforwardly determine the spectral exponent to be  $\tau = 1$  until we realise that  $\sum_{\omega \geq 1} \frac{1}{[\omega(\omega+1)]} = 1$  resulting in the divergence of the amplitude,  $A$ , for this choice of  $\tau$ . This surprising result suggests that we consider the more general divergence

$$F(x) \sim x^{-1} [\ln(1/x)]^\rho \text{ as } x \rightarrow 0. \quad (11)$$

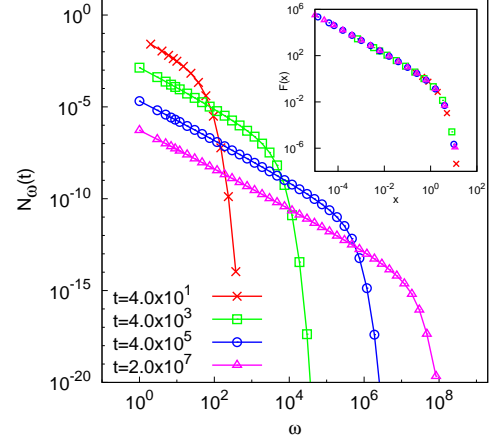


FIG. 3: Time evolution of wave spectrum in the decay case. The main panel shows snapshots of  $N_\omega(t)$  at a succession of times. The inset shows same data collapsed according to the scaling in Eq. (9).

where we have introduced a logarithmic spectral exponent,  $\rho$ , with which one may hope to cancel the divergence we have just encountered. The tail of the wave spectrum then has the form

$$N_\omega(t) = \frac{A}{s(t)} \frac{1}{\omega} \left[ \ln \left( \frac{s(t)}{\omega} \right) \right]^\rho \text{ for } \omega \ll s(t). \quad (12)$$

Setting  $k = 1$  in this formula gives the asymptotic form of  $n_1(t)$ . Substituting these formulae into Eq. (3) one finds that the leading term on the left hand side is of order  $s(t)^{-2} \ln[s(t)]^\rho$  and the leading order term on the right hand side is of order  $s(t)^{-2} \ln[s(t)]^{2\rho-1}$  (not  $s(t)^{-2} \ln[s(t)]^{2\rho}$  as one might naively expect owing to the cancellation alluded to above). Thus we should choose  $\rho = 1$  for the logarithmic spectral exponent so that the asymptotic form of the scaling function in the decay case is:

$$F(x) \sim x^{-1} \ln(1/x) \text{ as } x \rightarrow 0. \quad (13)$$

Fig. 4 shows the numerically obtained scaling function rescaled according to this formula. The plateau at small  $x$  provides strong numerical support for Eq. (13).

In principle, one should also obtain the amplitude at this point but this turns out to be easier using Eq. (2) for  $N(t)$  since certain sums which arise can be computed exactly in that case. The total wave action is

$$\begin{aligned} N(t) &= A s(t)^{-1} \sum_{\omega=1}^{s(t)} \omega^{-1} \ln \left[ \frac{s(t)}{\omega} \right] \\ &\approx A s(t)^{-1} \int_1^{s(t)} \int_{\omega=1}^{s(t)} \omega^{-1} \ln \left[ \frac{s(t)}{\omega} \right] d\omega \\ &= \frac{A \ln[s(t)]^2}{2 s(t)}. \end{aligned}$$

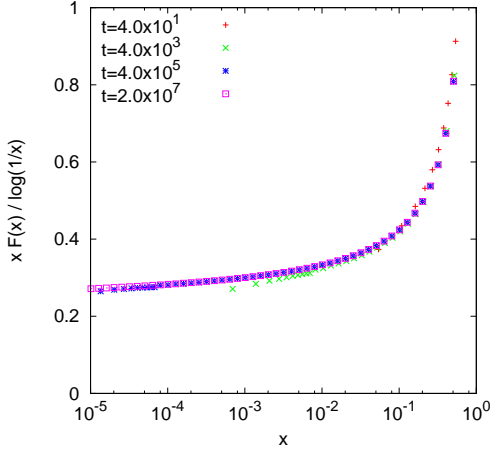


FIG. 4: Scaling function,  $F(x)$ , compensated for the theoretical small  $x$  divergence,  $x^{-1} \ln(1/x)$ , expected from Eq. (13).

Substituting this into the left-hand side of Eq. (2), and Eq. (12) into the right-hand side, and computing the leading terms we find the balance

$$-\frac{A s_0}{2} \frac{\ln[s(t)]^2}{s(t)^2} = -\frac{A^2}{2} \frac{\ln[s(t)]^2}{s(t)^2} \sum_{\omega=1}^{\infty} \frac{1}{\omega^2}.$$

The sum gives  $\pi^2/6$  from which we conclude that  $\frac{A}{s_0} = \frac{\pi^2}{6}$ . Recalling that  $s(t) \sim s_0 t$  this relation gives us the following nontrivial asymptotic decay laws for the total density and number of primary waves respectively:

$$N(t) \sim \frac{3}{\pi^2} \frac{(\ln t)^2}{t} \quad (14)$$

$$N_1(t) \sim \frac{6}{\pi^2} \frac{\ln t}{t}. \quad (15)$$

These predictions are validated numerically in Fig. 5.

To conclude, we have used the analogy between three-wave turbulence and aggregation-fragmentation processes to study analytically the decay kinetics of a simple wave turbulence model. We found that the kinetics have non-trivial scaling properties, even in this simple case, which differ significantly from the corresponding aggregation process. Our results suggest that decaying wave turbulence should be studied in greater detail than it has been to date.

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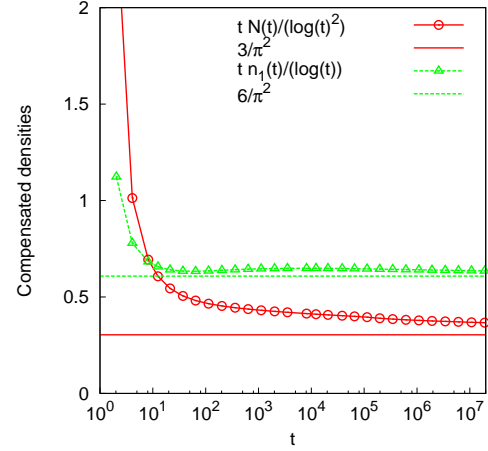


FIG. 5: Time decay of total wave action,  $N(t)$ , and primary wave action,  $N_1(t)$ , compensated by the theoretically predicted asymptotic decay rates taken from Eq. (14) and Eq.(15) respectively. The theoretically expected plateau values are attained to within a few percent accuracy.

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- [18] Discreteness means that frequencies are multiples of the primary frequency. The wave-action distribution is then  $N_\omega(t)$  with  $\omega = 1, 2, \dots$ . In the scaling limit, there is no difference between discrete and continuous.

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